Buckling of a Compression Member in a Rigid-Joint Truss. Part I: Equal End Restraints

Charles Libove*
Syracuse University, Syracuse, N. Y.

An approximate method is presented for checking the stability of a compression member in a rigid-joint welded-tubular truss which is symmetrical about a plane that perpendicularly bisects the compression member. The method is based on the assumption that the restraining members emanating from the ends of the compression member have ball-and-socket joints at their far ends. It is found that the compression member may, theoretically, buckle in either of two mutually perpendicular planes, and formulas are developed for determining these planes and the end restraint associated with each one, and the smaller end restraint is presumed to govern. With the end restraint known, equations or tables give the buckling load; comparison of this computed buckling load and the actual load in the compression member indicates whether or not the compression member is stable under its actual load. Numerical examples are included.

Introduction

THIS paper is intended as a contribution to the buckling analysis of rigid-joint welded-tubular trusses of the type used in the fuselage and engine mount of some aircraft. The truss is assumed to be loaded only at the joints, and its geometry is such, that if the rigid joints were replaced by ball-and-socket joints, the truss would still be a structure rather than a mechanism. In such a case it is customary to neglect the prebuckling deformations and to assume that during buckling the joints may rotate but do not displace, and this hypothesis is adopted here. Its accuracy is verified by Table 1 of Ref. 1.

It is well known that a rigid-joint truss is not divisible into buckled and unbuckled members. In the words of F. Bleich, "Because of the rigid connections . . . deflection of one member in the buckling state causes distortion of the other members of the structure. Each member is elastically restrained by the others, and the degree of restraint of any particular element depends upon the flexural rigidity and the axial loading of all other elements." (Ref. 2, p. 193.) Thus it is, strictly speaking, necessary to consider the entire truss as a single unit in determining the critical loading states, and methods of analysis are available for determining the buckling loads of rigid-joint trusses from this precise point of view.[1]

For trusses with many joints, however, the exact procedures, based on considering the entire truss as a unit, become rather complex. For example, if the joint rotations during buckling are used as unknowns, the exact approach leads to three times as many simultaneous homogeneous linear equations as there are joints, in the case of a space truss or out-of-plane buckling of a plane truss. The coefficients in these equations are functions of the tensions and compressions in the bars, which in turn are functions of the state of loading on the truss. The critical states are those for which the determinant of these coefficients vanishes. Simplified versions of the exact approach, avoiding the use of simultaneous equations, have been proposed by Lundquist[3,4] and Hoff[5] for planar buckling of plane trusses, but these still involve considerable calculation.

Because of the computational difficulties associated with exact methods, an approximate method is sometimes used for checking the stability of a plane truss against buckling in its plane. In this method one considers a small group of bars at a time, each group consisting of a selected compression member plus the members emanating from its ends, with the latter assumed to be hinged at their far ends. The rotational stiffness furnished to each end of the compression member by the members emanating from the ends is computed, taking into account the axial forces in these restraining members. With the rotational restraint known at each end of the compression member, its buckling load can be computed and compared with its actual compressive load. This procedure is carried out for each compression member, and the structure is considered safe against buckling if, in each case, the actual load in the compression member is less than its computed buckling load.

This approximate procedure is discussed in some detail in Ref. 2 (pp. 233–239 and 245–248). The chart of Ref. 6 and nomogram of Ref. 7 are useful means of obtaining the buckling load of a compression member, knowing the rotational stiffnesses of its end restraints. Plasticity can be accounted for by using an effective modulus (the tangent modulus is suggested) wherever the elastic theory calls for a Young's modulus.

The purpose of the present paper is to develop the corresponding procedure for checking the stability of compression members in a rigid-joint space truss or the stability against out-of-plane buckling of compression members in a rigid-joint plane truss. In this part, only the case of a compression member with restraints identical at both ends is considered; that is, the compression member under consideration and the bars emanating from its ends form a grouping which is symmetrical (in geometry and loading) about the plane that perpendicularly bisects the compression member. As a result of the symmetry and the assumed absence of continuity at the far ends of the restraining members, one can expect the buckled compression member to have a planar elastic curve and to be free of torsion. The more general case of unequal end restraints will be treated in Part II.

Basic Unit

In accordance with the prospectus previously outlined, the basic unit to be analyzed is the symmetrical group of members shown schematically in Fig. 1. It consists of the compression member AB under consideration and the restraining members $A1, A2, \ldots, B1, B2, \ldots$ adjoined to it. The axial force in each member is known, and it is required to determine the buckling load of member AB corresponding to the currently existing end restraints. The members are not necessarily coplanar, and even if coplanar, their plane is not necessarily the plane in which member AB will buckle. Each

Received August 2, 1965; revision received March 2, 1966.
* Associate Professor, Department of Mechanical and Aerospace Engineering. Member AIAA.

Table 1 **Relation between P_{cr} and m**

$\sqrt{\frac{P_{cr}L^2}{EI}}$	$\frac{mL}{4EI}$	$\sqrt{\frac{P_{cr}L^2}{EI}}$	$\frac{mL}{4EI}$	$\sqrt{\frac{P_{cr}L^2}{EI}}$	$\frac{mL}{4EI}$	$\sqrt{\frac{P_{cr}L^2}{EI}}$	$\frac{mL}{4EI}$	$\sqrt{\frac{P_{cr}L^2}{EI}}$	$\frac{mL}{4EI}$	$\sqrt{\frac{P_{cr}L^2}{EI}}$	$\frac{mL}{4EI}$	$\sqrt{\frac{P_{cr}L^2}{EI}}$	$\frac{mL}{4EI}$
0	-.500000	1.67	-.378009	2.44	-.223218	3.21	.027459	3.98	.443390	4.75	1.23302	5.52	3.43916
.1	-.499589	1.68	-.376468	2.45	-.220667	3.22	.031575	3.99	.450495	4.76	1.24805	5.53	3.49587
.2	-.498332	1.69	-.374916	2.46	-.218099	3.23	.035718	4.00	.457658	4.77	1.26326	5.54	3.55405
.3	-.496244	1.70	-.373353	2.47	-.215516	3.24	.039887	4.01	.464878	4.78	1.27866	5.55	3.61377
.4	-.493315	1.71	-.371779	2.48	-.212917	3.25	.044084	4.02	.472157	4.79	1.29424	5.56	3.67509
.5	-.489540	1.72	-.370193	2.49	-.210302	3.26	.048307	4.03	.479496	4.80	1.31002	5.57	3.73808
.6	-.484909	1.73	-.368596	2.50	-.207671	3.27	.052559	4.04	.486894	4.81	1.32600	5.58	3.80281
.7	-.479415	1.74	-.366988	2.51	-.205023	3.28	.056838	4.05	.494353	4.82	1.34219	5.59	3.86935
.8	-.473044	1.75	-.365368	2.52	-.202360	3.29	.061145	4.06	.501874	4.83	1.35857	5.60	3.93779
.9	-.465785	1.76	-.363737	2.53	-.199679	3.30	.065480	4.07	.509457	4.84	1.37517	5.61	4.00820
1.0	-.457622	1.77	-.362095	2.54	-.196983	3.31	.069844	4.08	.517104	4.85	1.39199	5.62	4.08068
1.01	-.456755	1.78	-.360441	2.55	-.194269	3.32	.074236	4.09	.524814	4.86	1.40902	5.63	4.15532
1.02	-.455879	1.79	-.358775	2.56	-.191539	3.33	.078657	4.10	.532589	4.87	1.42628	5.64	4.23223
1.03	-.454994	1.80	-.357098	2.57	-.188793	3.34	.083108	4.11	.540430	4.88	1.44376	5.65	4.31150
1.04	-.454099	1.81	-.355409	2.58	-.186029	3.35	.087588	4.12	.548337	4.89	1.46148	5.66	4.39326
1.05	-.453196	1.82	-.353709	2.59	-.183248	3.36	.092097	4.13	.556311	4.90	1.47944	5.67	4.47762
1.06	-.452282	1.83	-.351996	2.60	-.180450	3.37	.096637	4.14	.564353	4.91	1.49764	5.68	4.56471
1.07	-.451360	1.84	-.350272	2.61	-.177635	3.38	.101207	4.15	.572465	4.92	1.51609	5.69	4.65467
1.08	-.450428	1.85	-.348537	2.62	-.174803	3.39	.105807	4.16	.580646	4.93	1.53479	5.70	4.74766
1.09	-.449487	1.86	-.346789	2.63	-.171953	3.40	.110438	4.17	.588899	4.94	1.55375	5.71	4.84382
1.10	-.448536	1.87	-.345029	2.64	-.169086	3.41	.115100	4.18	.597223	4.95	1.57298	5.72	4.94331
1.11	-.447576	1.88	-.343257	2.65	-.166201	3.42	.119794	4.19	.605620	4.96	1.59248	5.73	5.04634
1.12	-.446607	1.89	-.341474	2.66	-.163298	3.43	.124519	4.20	.614090	4.97	1.61226	5.74	5.15309
1.13	-.445628	1.90	-.339678	2.67	-.160377	3.44	.129276	4.21	.622636	4.98	1.63231	5.75	5.26377
1.14	-.444640	1.91	-.337870	2.68	-.157439	3.45	.134065	4.22	.631257	4.99	1.65266	5.76	5.37860
1.15	-.443642	1.92	-.336050	2.69	-.154482	3.46	.138887	4.23	.639955	5.00	1.67331	5.77	5.49783
1.16	-.442634	1.93	-.334218	2.70	-.151508	3.47	.143741	4.24	.648731	5.01	1.69426	5.78	5.62171
1.17	-.441617	1.94	-.332374	2.71	-.148515	3.48	.148628	4.25	.657586	5.02	1.71552	5.79	5.75054
1.18	-.440591	1.95	-.330517	2.72	-.145503	3.49	.153549	4.26	.666520	5.03	1.73710	5.80	5.88461
1.19	-.439555	1.96	-.328648	2.73	-.142473	3.50	.158504	4.27	.675536	5.04	1.75900	5.81	6.02427
1.20	-.438509	1.97	-.326766	2.74	-.139424	3.51	.163492	4.28	.684635	5.05	1.78124	5.82	6.16985
1.21	-.437453	1.98	-.324872	2.75	-.136357	3.52	.168515	4.29	.693817	5.06	1.80381	5.83	6.32179
1.22	-.436388	1.99	-.322965	2.76	-.133271	3.53	.173572	4.30	.703083	5.07	1.82674	5.84	6.48047
1.23	-.435314	2.00	-.321046	2.77	-.130165	3.54	.178665	4.31	.712436	5.08	1.85002	5.85	6.64638
1.24	-.434229	2.01	-.319115	2.78	-.127041	3.55	.183793	4.32	.721875	5.09	1.87368	5.86	6.82004
1.25	-.433135	2.02	-.317170	2.79	-.123897	3.56	.188956	4.33	.731403	5.10	1.89770	5.87	7.00200
1.26	-.432031	2.03	-.315213	2.80	-.120734	3.57	.194155	4.34	.741021	5.11	1.92212	5.88	7.19289
1.27	-.430917	2.04	-.313243	2.81	-.117551	3.58	.199391	4.35	.750729	5.12	1.94692	5.89	7.39336
1.28	-.429793	2.05	-.311260	2.82	-.114349	3.59	.204664	4.36	.760530	5.13	1.97214	5.90	7.60419
1.29	-.428660	2.06	-.309264	2.83	-.111126	3.60	.209973	4.37	.770425	5.14	1.99777	5.91	7.82621
1.30	-.427517	2.07	-.307255	2.84	-.107884	3.61	.215320	4.38	.780415	5.15	2.02382	5.92	8.06032
1.31	-.426363	2.08	-.305233	2.85	-.104622	3.62	.220705	4.39	.790501	5.16	2.05032	5.93	8.30758
1.32	-.425200	2.09	-.303198	2.86	-.101340	3.63	.226128	4.40	.800685	5.17	2.07726	5.94	8.56911
1.33	-.424027	2.10	-.301150	2.87	-.098037	3.64	.231589	4.41	.810969	5.18	2.10467	5.95	8.84622
1.34	-.422844	2.11	-.299089	2.88	-.094714	3.65	.237090	4.42	.821354	5.19	2.13255	5.96	9.14033
1.35	-.421651	2.12	-.297014	2.89	-.091370	3.66	.242630	4.43	.831841	5.20	2.16092	5.97	9.45307
1.36	-.420448	2.13	-.294927	2.90	-.088005	3.67	.248209	4.44	.842433	5.21	2.18978	5.98	9.78628
1.37	-.419235	2.14	-.292825	2.91	-.084620	3.68	.253829	4.45	.853130	5.22	2.21917	5.99	10.1421
1.38	-.418012	2.15	-.290710	2.92	-.081213	3.69	.259489	4.46	.863935	5.23	2.24908	6.00	10.5229
1.39	-.416778	2.16	-.288582	2.93	-.077786	3.70	.265190	4.47	.874849	5.24	2.27954	6.01	10.9314
1.40	-.415535	2.17	-.286440	2.94	-.074337	3.71	.270933	4.48	.885874	5.25	2.31055	6.02	11.3707
1.41	-.414281	2.18	-.284285	2.95	-.070866	3.72	.276718	4.49	.897011	5.26	2.34214	6.03	11.8446
1.42	-.413017	2.19	-.282116	2.96	-.067374	3.73	.282544	4.50	.908263	5.27	2.37433	6.04	12.3573
1.43	-.411743	2.20	-.279932	2.97	-.063860	3.74	.288414	4.51	.919632	5.28	2.40712	6.05	12.9137
1.44	-.410459	2.21	-.277736	2.98	-.060324	3.75	.294326	4.52	.931118	5.29	2.44055	6.06	13.5197
1.45	-.409164	2.22	-.275525	2.99	-.056766	3.76	.300283	4.53	.942725	5.30	2.47462	6.07	14.1825
1.46	-.407859	2.23	-.273300	3.00	-.053186	3.77	.306283	4.54	.954453	5.31	2.50936	6.08	14.9102
1.47	-.406543	2.24	-.271061	3.01	-.049583	3.78	.312328	4.55	.966306	5.32	2.54479	6.09	15.7130
1.48	-.405218	2.25	-.268808	3.02	-.045957	3.79	.318418	4.56	.978285	5.33	2.58093	6.10	16.6031
1.49	-.403881	2.26	-.266541	3.03	-.042309	3.80	.324554	4.57	.990393	5.34	2.61780	6.11	17.5960
1.50	-.402535	2.27	-.264259	3.04	-.038638	3.81	.330735	4.58	1.00263	5.35	2.65542	6.12	18.7101
1.51	-.401178	2.28	-.261964	3.05	-.034944	3.82	.336963	4.59	1.01500	5.36	2.69383	6.13	19.9695
1.52	-.399810	2.29	-.259653	3.06	-.031226	3.83	.343239	4.60	1.02751	5.37	2.73303	6.14	21.4041
1.53	-.398432	2.30	-.257329	3.07	-.027485	3.84	.349561	4.61	1.04015	5.38	2.77308	6.15	23.0540
1.54	-.397043	2.31	-.254989	3.08	-.023720	3.85	.355932	4.62	1.05293	5.39	2.81398	6.16	24.9714
1.55	-.395644	2.32	-.252635	3.09	-.019932	3.86	.362352	4.63	1.06586	5.40	2.85577	6.17	27.2271
1.56	-.394233	2.33	-.250267	3.10	-.016119	3.87	.368820	4.64	1.07893	5.41	2.89848	6.18	29.9195
1.57	-.392813	2.34	-.247883	3.11	-.012283	3.88	.375339	4.65	1.09214	5.42	2.94213	6.19	33.1893
1.58	-.391381	2.35	-.245485	3.12	-.008421	3.89	.381908	4.66	1.10551	5.43	2.98678	6.20	37.2447
1.59	-.389939	2.36	-.243072	3.13	-.004536	3.90	.388527	4.67	1.11903	5.44	3.03244	6.21	42.4076
1.60	-.388486	2.37	-.240644	3.14	-.000625	3.91	.395199	4.68	1.13271	5.45	3.07915	6.22	49.2039
1.61	-.387022	2.38	-.238200	3.15	.003310	3.92	.401922	4.69	1.14654	5.46	3.12696	6.23	58.5550
1.62	-.385547	2.39	-.235742	3.16	.007271	3.93	.408698	4.70	1.16053	5.47	3.17590	6.24	72.2356
1.63	-.384061	2.40	-.233268	3.17	.011257	3.94	.415527	4.71	1.17469	5.48	3.22602	6.25	94.1596
1.64	-.382565	2.41	-.230779	3.18	.015269	3.95	.422410	4.72	1.18902	5.49	3.27736	6.26	134.993
1.65	-.381057	2.42	-.228274	3.19	.019306	3.96	.429348	4.73	1.20351	5.50	3.32996	6.27	237.76
1.66	-.379539	2.43	-.225754	3.20	.023370	3.97	.436341	4.74	1.21818	5.51	3.38388	6.28	985.9

bar is uniform and has a cross section with equal moments of inertia about all centroidal axes. Not only the geometry but also the bar forces are symmetrical about the plane of symmetry shown in Fig. 1.

Rigid joints are assumed at the ends A and B of the compression member, whereas ball-and-socket joints are assumed at the far ends 1, 2, . . . of the restraining members. All the joints are assumed to be held against displacement during buckling. The assumption of ball-and-socket joints at the far ends of the restraining members is tantamount to neglecting the torsional stiffness of the restraining members. This neglect is conservative, since the torsional stiffness of a restraining member may be partly or entirely ineffective because of the unknown twisting rotation of its far end. On the other hand, the flexural stiffness of a restraining member always will be effective as long as the far end is prevented from deflecting.

An xyz right-hand Cartesian coordinate system will be used in the subsequent analysis, with origin at A and x axis directed toward B. The y and z axes, not shown in Fig. 1, may have any convenient orientation in the plane through A perpendicular to AB. The direction cosines of the restraining members must be known. The symbols l_i, m_i, and n_i will denote the known direction cosines of the restraining member Ai ($i = 1, 2, \ldots$) relative to the x, y, and z axes, respectively. The symbols $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ will denote the unit vectors associated with the x, y, and z axes, respectively.

It may be that the direction cosines l_i', m_i', and n_i' of member Ai already are known with respect to some other right-hand system of axes x', y', and z', none of which is parallel to member AB. These can be converted to direction cosines relative to the x, y, and z axes through the transformation

$$\begin{Bmatrix} l_i \\ m_i \\ n_i \end{Bmatrix} = \begin{bmatrix} a_{xx'} & a_{xy'} & a_{xz'} \\ a_{yx'} & a_{yy'} & a_{yz'} \\ a_{zx'} & a_{zy'} & a_{zz'} \end{bmatrix} \begin{Bmatrix} l_i' \\ m_i' \\ n_i' \end{Bmatrix} \tag{1}$$

where each element of the square matrix is the cosine of the angle between the axes indicated by its subscripts.

Equation (1) assumes that an xyz reference frame already has been constructed for the compression member under consideration. There are many choices for this reference frame since only the direction of its x axis is fixed. To obtain one suitable reference frame from the many, we may select the one whose y axis is perpendicular to both the x and x' axes. The direction cosines of such a y axis are defined by the following equations:

$$a_{yx'} = 0 \qquad a_{yy'}a_{xy'} + a_{yz'}a_{xz'} = 0$$
$$a_{yy'}^2 + a_{yz'}^2 = 1 \tag{2}$$

Having solved for a set of values of $a_{yx'}(= 0)$, $a_{yy'}$, and $a_{yz'}$ from Eqs. (2), one can determine the orientation of the remaining axis, the z axis, from the relationships

$$\mathbf{k} = \mathbf{i} \times \mathbf{j} \tag{3}$$

$$\left.\begin{aligned} \mathbf{i} &= \mathbf{i}'\, a_{xx'} + \mathbf{j}'\, a_{xy'} + \mathbf{k}'\, a_{xz'} \\ \mathbf{j} &= \mathbf{i}'\, a_{yx'} + \mathbf{j}'\, a_{yy'} + \mathbf{k}'\, a_{yz'} \\ \mathbf{k} &= \mathbf{i}'\, a_{zx'} + \mathbf{j}'\, a_{zy'} + \mathbf{k}'\, a_{zz'} \end{aligned}\right\} \tag{4}$$

where $\mathbf{i}'$, $\mathbf{j}'$, and $\mathbf{k}'$ are the unit vectors associated with the x', y', and z' axes, respectively. If Eqs. (4) are used to eliminate $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in Eq. (3), the resulting vector equation is equivalent to the following three scalar equations:

$$\left.\begin{aligned} a_{zx'} &= a_{xy'}a_{yz'} - a_{yy'}a_{xz'} \\ a_{zy'} &= a_{xz'}a_{yx'} - a_{yz'}a_{xx'} \\ a_{zz'} &= a_{xx'}a_{yy'} - a_{yx'}a_{xy'} \end{aligned}\right\} \tag{5}$$

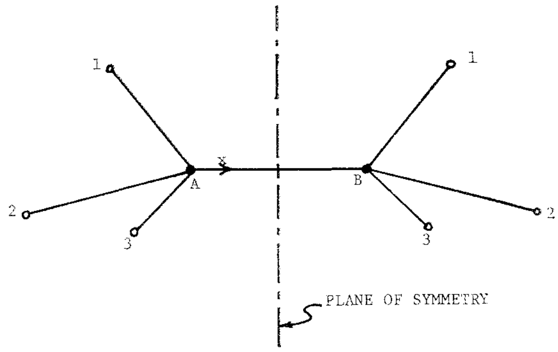

Fig. 1 Symmetrical group of bars, consisting of the compression member *AB* and the restraining members emanating from its ends.

or, taking account of the first of Eqs. (2),

$$a_{zx'} = a_{xy'}a_{yz'} - a_{yy'}a_{xz'} \qquad a_{zy'} = -a_{yz'}a_{xx'}$$
$$a_{zz'} = a_{xx'}a_{yy'} \tag{6}$$

Thus all the elements of the square matrix in Eq. (1) have been fixed, those in the first line by virtue of the known orientation of AB in the $x'y'z'$ reference frame, those in the second line by virtue of Eqs. (2), and those in the third line by Eqs. (6).

Relation between End Restraint and Buckling Load

It can be expected that during buckling, symmetry is preserved and the originally straight axis of member AB deflects into a curve that lies in a single plane. This plane, hereinafter called the plane of buckling, passes through points A and B, and its orientation can be determined from the requirement that the bending moment in an end of the buckled compression member and the resisting moment produced by the cluster of restraining members emanating from that end must be coplanar.

The details of the determination of the plane of buckling and the calculation of the resisting moment are presented in the next section. For the time being, let us assume that the plane of buckling is known and that the stiffness m of the cluster of restraining members at either end is known, where m is defined as the resisting moment produced by the restraining members per radian of rotation of the end of the compression member in its plane of buckling. This definition, in equation form, is

$$m = M/\theta \tag{7}$$

where θ is the infinitesimal angle of rotation (in radians) of the end of the buckled compression member, measured in the plane of buckling, and M is the corresponding resisting moment developed by the restraining members. The angle θ may be more precisely defined as the angle between the straight line AB and the tangent to the elastic curve of the buckled compression member at end A.

If m is known, the buckling load of the compression member readily can be determined, or its stability under the load that it is required to carry easily can be checked, for the compression member is, in effect, a planarly buckled column with ends equally elastically restrained against rotation, and the relationship between the buckling load P_{cr} of such a column and the stiffness m of the end restraint is well known and has been published in many forms. In Ref. 8 the relationship is given as

$$P_{cr} = c\pi^2 \bar{E}I/L^2 \tag{8}$$

where $\bar{E}$ is the effective modulus of the column, herein to be

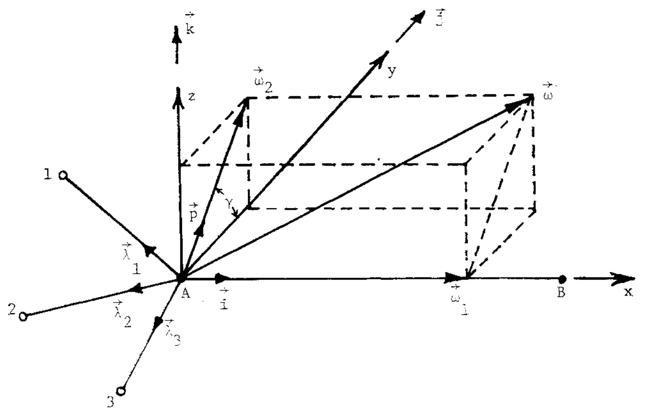

Fig. 2 Diagram showing rotation vector ω of joint A and its components.

taken as the tangent modulus, I is the moment of inertia of its cross section about a centroidal axis perpendicular to the plane of buckling, L is the length of the column (the distance AB in the present case), and c is a fixity coefficient related to m through the equation

$$\frac{mL}{\bar{E}I} = \frac{-\pi(c)^{1/2}}{\tan[\pi(c/4)^{1/2}]} = \frac{-(P_{cr}L^2/\bar{E}I)^{1/2}}{\tan(P_{cr}L^2/4\bar{E}I)^{1/2}} \tag{9}$$

Table 1, obtained from Ref. 9, gives the relationship between m and P_{cr} in numerical form. Table 1 can be used in two ways: 1) with m known, it gives P_{cr} for comparison with the actual load P that the compression member has to carry, and 2) with P_{cr} set equal to P, it gives the stiffness m_{cr} for which the actual load P would be a buckling load; comparison of this stiffness with the actually stiffness m indicates whether or not the compression member is stable under its load P. For P beyond the range of Table 1 stability is not possible.

Determination of Plane of Buckling and End Restraint

Regarding the joint at A as a small rigid body, we may represent its infinitesimal rotation during buckling by means of a vector in accordance with the right-hand rule. This is the vector $\boldsymbol{\omega}$ shown in Fig. 2. This rotation vector can be resolved into two components: $\boldsymbol{\omega}_1$ along AB, and $\boldsymbol{\omega}_2$ perpendicular to AB (i.e., in the yz plane) and making some angle γ with respect to the y axis. The vector $\boldsymbol{\omega}_1$ represents a twisting rotation of end A of the compression member, but no torque is developed in the compressing member as a result of it, because the other end B undergoes exactly the same twisting rotation. On the other hand, the vector $\boldsymbol{\omega}_2$ represents a bending rotation of end A of the compression member. As a result of $\boldsymbol{\omega}_2$ and the corresponding rotation at the other end, a bending moment will, in general, be produced in the compression member at end A, and the vector representing this bending moment will coincide with $\boldsymbol{\omega}_2$. The plane of buckling is perpendicular to the vector $\boldsymbol{\omega}_2$; hence its orientation is defined by the angle γ.

Introducing the unit vector $\mathbf{p}$ in the direction of $\boldsymbol{\omega}_2$ (see Fig. 2), we may write the rotation vector $\boldsymbol{\omega}$ as

$$\begin{aligned}\boldsymbol{\omega} &= \mathbf{i}\phi + \mathbf{p}\theta \\ &= \mathbf{i}\phi + \mathbf{j}\theta\cos\gamma + \mathbf{k}\theta\sin\gamma\end{aligned} \tag{10}$$

where ϕ and θ are the twisting and bending angles of rotation, respectively, of end A of the compression member, and $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the previously introduced unit vectors.

The rotation $\boldsymbol{\omega}$ of joint A also produces bending moments in the restraining members. As an aid in computing these bending moments, let us introduce the unit vectors $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \ldots$ directed along the restraining members $A1, A2, \ldots,$ respectively, as shown in Fig. 2. Considering the typical restraining member Ai ($i = 1, 2, \ldots$), we note that $\boldsymbol{\omega}$ can be resolved into a component $\boldsymbol{\omega}_1'$ along the member Ai and a component $\boldsymbol{\omega}_2'$ perpendicular to it, just as it was resolved into similar components in relation to member AB. This decomposition is shown in Fig. 3. The vector $\boldsymbol{\omega}_1'$ represents a twisting rotation of end A of the restraining member Ai, which, however, produces no torque in member Ai because of the assumed ball-and-socket joint at i. The vector $\boldsymbol{\omega}_2'$, which represents a bending rotation of end A of member Ai, lies in the plane of Ai and $\boldsymbol{\omega}$ and has a magnitude of $|\boldsymbol{\omega}| \sin\beta$, where β is the true angle between Ai and $\boldsymbol{\omega}$. The magnitude of the vector $\boldsymbol{\omega}_2'$ is therefore correctly given by $\boldsymbol{\lambda}_i \times \boldsymbol{\omega}$, but not the direction of $\boldsymbol{\omega}_2'$. The vector $\boldsymbol{\lambda}_i \times \boldsymbol{\omega}$ would have to be rotated 90° about Ai in order to coincide with $\boldsymbol{\omega}_2'$. The required rotation can be accomplished by a vector multiplication of $\boldsymbol{\lambda}_i \times \boldsymbol{\omega}$ into $\boldsymbol{\lambda}_i$. Hence

$$\boldsymbol{\omega}_2' = (\boldsymbol{\lambda}_i \times \boldsymbol{\omega}) \times \boldsymbol{\lambda}_i \tag{11}$$

The vector-triple-product formula transforms this to

$$\boldsymbol{\omega}_2' = \boldsymbol{\omega} - \boldsymbol{\lambda}_i(\boldsymbol{\omega}\cdot\boldsymbol{\lambda}_i) \tag{12}$$

The bending rotation $\boldsymbol{\omega}_2'$ induces the following bending moment in the end of the restraining member Ai:

$$\mathbf{M}_i = K_i\,\boldsymbol{\omega}_2' = K_i[\boldsymbol{\omega} - \boldsymbol{\lambda}_i(\boldsymbol{\omega}\cdot\boldsymbol{\lambda}_i)] \tag{13}$$

where K_i is the stiffness (moment per radian) of the restraining member with its far end hinged. If the restraining member carries no axial load, its K_i equals $3E_iI_i/L_i$, where E_i, I_i, and L_i are its Young's modulus, moment of inertia, and length, respectively. If the restraining member is axially loaded, then K_i should be computed with beam-column and plasticity effects taken into account, the latter based on the tangent modulus for conservatism. Reference 9, in which S^{II} corresponds to one-fourth of K_i, may be used for this purpose, as may a number of other references.

The total moment $\mathbf{M}$ induced in the cluster of restraining members at A is obtained by summing Eq. (13) with respect to i:

$$\mathbf{M} = \sum_i K_i[\boldsymbol{\omega} - \boldsymbol{\lambda}_i(\boldsymbol{\omega}\cdot\boldsymbol{\lambda}_i)] \tag{14}$$

The as-yet-unknown angle γ, which defines the location of the plane of buckling, and the unknown ratio of ϕ to θ can be determined from the condition that the bending moment in end of the compression member, and the resultant moment

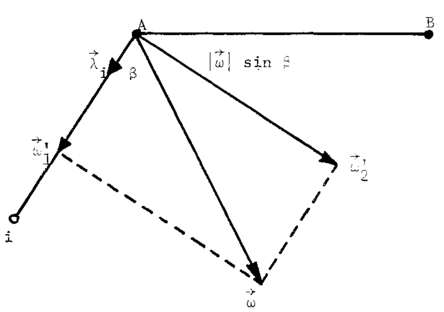

Fig. 3 Resolution of ω vector into components parallel and perpendicular to member Ai.

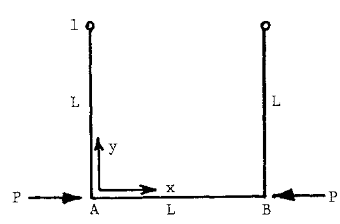

Fig. 4 Configuration for Example 2.

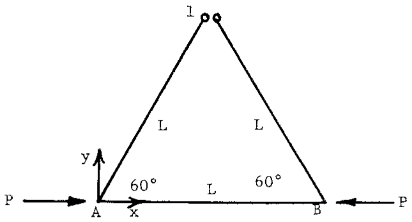

Fig. 5 Configuration for Example 3.

$\mathbf{M}$ in the restraining members must be coplanar; i.e., the cross product of the vectors representing these two moments must vanish, which implies that

$$\mathbf{p} \times \mathbf{M} = 0 \tag{15}$$

or, with $\mathbf{M}$ eliminated through Eq. (14),

$$\sum_i K_i[\mathbf{p} \times \boldsymbol{\omega} - (\mathbf{p} \times \boldsymbol{\lambda}_i)(\boldsymbol{\omega}\cdot\boldsymbol{\lambda}_i)] = 0 \tag{16}$$

In Eq. (16), $\boldsymbol{\omega}$ may be replaced by its expression from Eq. (10), $\mathbf{p}$ by $\mathbf{j}\cos\gamma + \mathbf{k}\sin\gamma$, and $\boldsymbol{\lambda}_i$ by its formula in terms of the Cartesian unit vectors, i.e.,

$$\boldsymbol{\lambda}_i = \mathbf{i}\, l_i + \mathbf{j}\, m_i + \mathbf{k}\, n_i \tag{17}$$

With these substitutions made, Eq. (16) becomes

$$-\mathbf{i}\theta \sum_i K_i\left[\left(\frac{\phi}{\theta}\right)l_i + m_i\cos\gamma + n_i\sin\gamma\right] \times (n_i\cos\gamma - m_i\sin\gamma) + \mathbf{j}\theta\sum_i K_i\left[\left(\frac{\phi}{\theta}\right) - l_i\left\{\left(\frac{\phi}{\theta}\right)l_i + m_i\cos\gamma + n_i\sin\gamma\right\}\right]\sin\gamma - \mathbf{k}\theta\sum_i K_i\left[\left(\frac{\phi}{\theta}\right) - l_i\left\{\left(\frac{\phi}{\theta}\right)l_i + m_i\cos\gamma + n_i\sin\gamma\right\}\right]\cos\gamma = 0 \tag{18}$$

Equation (18) implies three scalar equations, of which the last is seen to be redundant. The two remaining equations, which constitute a pair of simultaneous equations in γ and ϕ/θ, are

$$(\phi/\theta)(S_2\cos\gamma - S_1\sin\gamma) + S_3(\cos^2\gamma - \sin^2\gamma) + S_4\sin\gamma\cos\gamma = 0 \tag{19}$$

$$(\phi/\theta)S_5 - S_1\cos\gamma - S_2\sin\gamma = 0 \tag{20}$$

where

$$S_1 = \sum_i K_i l_i m_i \qquad S_2 = \sum_i K_i l_i n_i$$
$$S_3 = \sum_i K_i m_i n_i \qquad S_4 = \sum_i K_i(n_i^2 - m_i^2) \tag{21}$$
$$S_5 = \sum_i K_i(1 - l_i^2)$$

Solving Eq. (20) for ϕ/θ and eliminating ϕ/θ in Eq. (19) give

$$\phi/\theta = (S_1/S_5)\cos\gamma + (S_2/S_5)\sin\gamma \tag{22}$$

$$\tan 2\gamma = 2(S_1S_2 + S_3S_5)/(S_1^2 - S_2^2 - S_4S_5) \tag{23}$$

Equation (23) defines two possible values for γ differing by 90°, which indicates that there are two theoretically possible planes of buckling at right angles to each other. For each γ the ratio of twisting to bending rotation for the end of the compression member can be found from Eq. (22).

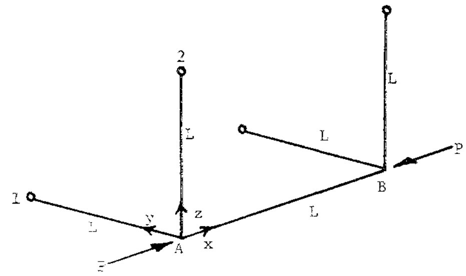

Fig. 6 Configuration for Example 4.

The stiffness m of the cluster of restraining members at A corresponding to either value of γ now can be obtained from Eq. (14) with $\boldsymbol{\omega}$ and $\boldsymbol{\lambda}_i$ expressed in $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form through Eqs. (10) and (17). It gives

$$\mathbf{M}/\theta = \mathbf{i}[(\phi/\theta)S_5 - S_1\cos\gamma - S_2\sin\gamma] + \mathbf{j}[S_6\cos\gamma - S_3\sin\gamma - (\phi/\theta)S_1] + \mathbf{k}[S_7\sin\gamma - S_3\cos\gamma - (\phi/\theta)S_2] \tag{24}$$

where

$$S_6 = \sum_i K_i(1 - m_i^2) \qquad S_7 = \sum_i K_i(1 - n_i^2) \tag{25}$$

The coefficient of $\mathbf{i}$ in Eq. (24) vanishes by virtue of Eq. (19), reflecting the absence of torque in the compression member during buckling. Using Eq. (19) to eliminate ϕ/θ in the remaining terms, one obtains

$$\mathbf{M}/\theta = \mathbf{j}\{[S_6 - (S_1^2/S_5)]\cos\gamma - [S_3 + (S_1S_2/S_5)]\sin\gamma\} + \mathbf{k}\{[S_7 - (S_2^2/S_5)]\sin\gamma - [S_3 + (S_1S_2/S_5)]\cos\gamma\} \tag{26}$$

from which

$$m \equiv |\mathbf{M}|/\theta = (m_y^2 + m_z^2)^{1/2} \tag{27}$$

with

$$\begin{aligned} m_y &\equiv [S_6 - (S_1^2/S_5)]\cos\gamma - [S_3 + (S_1S_2/S_5)]\sin\gamma \\ m_z &\equiv [S_7 - (S_2^2/S_5)]\sin\gamma - [S_3 + (S_1S_2/S_5)]\cos\gamma \end{aligned} \tag{28}$$

Procedure for Checking the Stability of the Compression Member

The procedure for checking the stability of the compression member AB can now be summarized as follows:

1) From Eq. (23) compute the two values of γ defining the normals to the two theoretically possible planes of buckling.

2) For each γ, compute m by means of Eq. (27); the value of γ corresponding to the smaller (less positive or more negative) value of m determines the actual plane of buckling.

3) Enter the smaller m in Table 1 to determine the corresponding value of P_{cr} for the compression member, and compare P_{cr} with the actual compressive force P in the member. If $P < P_{cr}$ the member is stable; if $P > P_{cr}$ the member is unstable.

Alternate step 3): Enter the actual P as P_{cr} in Table 1 and thus determine the value of m required to make this P the buckling load; denote this value of m by m_{cr}.

Table 2 Data for Example 1

Bar	Length, L, in.	Area, A, in.^{2}	I, in.^{4}	Force, P, lbs	$\bar{E} \times 10^{-6}$, psi	$L(P/\bar{E}I)^{1/2}$	K, in.-lb/rad	Direction cosines l	m	n
AB	18.0	0.06487	0.002833	3726	18.95	4.75	...	1	0	0
$A1$	12.8	0.06487	0.002833	3505	21.8	3.06	1172	−0.9363	0.3511	0
$A2$	17.7	0.06487	0.002833	1308	29.0	1.655	11150	−0.3381	−0.9411	0
$A3$	16.5	0.05252	0.002345	0	29.0	0	12380	0.4523	−0.8393	0.3016

Table 3 Calculation of $S_1, S_2, \ldots, S_7$ for Example 1

Bar	i	Klm	Kln	Kmn	$K(n^2 - m^2)$	$K(1 - l^2)$	$K(1 - m^2)$	$K(1 - n^2)$
$A1$	1	−385	0	0	−144	144	1027	1172
$A2$	2	3548	0	0	−9876	9876	1274	11150
$A3$	3	−4699	1689	−3133	−7594	9847	3660	11253
	$\sum_i$	−1536 (S_1)	1689 (S_2)	−3133 (S_3)	−17614 (S_4)	19867 (S_5)	5961 (S_6)	23575 (S_7)

If m_{cr} is less than the smaller of the two m values obtained in Step 2, the compression member is stable. If m_{cr} is greater than the smaller of these two m values, the compression member is unstable.

Examples

The first of the following illustrative applications is based on a possible design situation. The rest are idealized problems, selected because their solutions give further insight into the buckling of elastically restrained compression members.

Example 1

In some small aircraft the right and left fuselage trusses are connected by a welded-tubular truss-work, of the type shown in Fig. 1, going laterally across the cabin at about the dashboard location. Table 2 gives the geometry of an assumed design and the computed compression forces in the members under a particular ultimate loading condition. The material is AISI 4130 steel in condition "N" with an ultimate tensile strength of 95 ksi.

From the direction cosines l, m, and n of the bars relative to the x, y, and z axes, respectively, it is seen that in this example bars AB, $A1$ and $A2$ lie in one plane and member $A3$ is skew to this plane. The effective moduli $\bar{E}$ shown in Table 2 were obtained from the compressive stress P/A and the column curve of Ref. 10 for 4130 alloy-steel round tubing by the method suggested in Ref. 11. (By this method the plasticity reduction factor η is the ratio of the actual stress P/A to the Euler buckling stress of a column having P/A for its plastic buckling stress. If the plastic-buckling column curve agrees with the tangent-modulus theory, then $\bar{E}$ computed this way will be the tangent modulus associated with the stress P/A.) The quantity $L(P/\bar{E}I)^{1/2}$ in Table 2 corresponds to the quantity called $(L/j)_{eff}$ in Ref. 9; it was used in conjunction with the tables of that reference to compute the stiffnesses K, which are shown in Table 2. The stiffness K of a member is four times the stiffness S^{II} of Ref. 9.

The problem is to check the stability of member AB (actually, of the entire group) under the given loading. Table 3 shows the calculation of the sums S_1–S_7 needed to compute γ [Eq. (23)] and m [Eqs. (27) and (28)]. (In the column headings of Table 3, the subscript i has been omitted from the symbols K, l, m, and n for simplicity; the numerical values of these quantities were taken from Table 2.) With S_1–S_7 known, Eq. (23) gives $\tan 2\gamma = -0.37109$, whence the two theoretically possible values of γ are $-10.18°$ and $79.82°$. Equations (28) and (27) then give $m = 5255$ in.-lb/radian for $\gamma = -10.18°$ and $m = 24{,}060$ in.-lb/radian for $\gamma = 79.82°$. The smaller m determines the actual plane of buckling; it gives the following nondimensional elastic restraint for the compression member:

$$\frac{1}{4}\frac{mL}{\bar{E}I} = \frac{1}{4}\frac{(5255)(18.0)}{(18.95)10^6(0.002833)} = 0.4405$$

Entering this in Table 1, we obtain $(P_{cr}L^2/\bar{E}I)^{1/2} = 3.976$, whence

$$P_{cr} = \frac{(3.976)^2(18.95)10^6(0.002833)}{(18.0)^2} = 2620 \text{ lb}$$

Comparison of this with the actual compressive force of 3726 lb that member AB is required to carry leads to the conclusion that member AB probably would have buckled long before developing its required 3726-lb load.

We cannot conclude, of course, that the true buckling strength of member AB is the 2620-lb load previously computed, for when AB is carrying only 2620 lb, the forces in the restraining members $A1$, $A2$, and $A3$ will be only 2620/3726 of the P values shown in Table 2, thus altering all the data on which the calculation of P_{cr} was based. On physical grounds it can be argued, however, that the true P_{cr} of a compression member lies somewhere between the compressive force P for which the stability check is made and the computed P_{cr} to which the stability check leads. Applied to this example, the argument implies that the true P_{cr} is less than the required strength of 3726 lb, but more than the computed P_{cr} of 2620 lb.

If the true P_{cr} is desired, one readily can devise an iterative or trial-and-error procedure for obtaining it. Essentially, such a procedure would consist of making stability checks for a sequence of assumed values of P until the computed P_{cr} agrees with the assumed P. The assumed P for which this occurs would be the true P_{cr}.

Example 2

In this example the coplanar three-bar structure of Fig. 4 is considered. The structure lies in the xy plane, which is the plane of the paper; the z axis is directed upward, out of the paper. All bars have the same length L, Young's modulus E, and moment of inertia I. The central bar AB is loaded with a compressive force P, while the restraining bar at each end is unstressed. The problem is to determine the buckling load P_{cr} of the compression member, assuming perfectly elastic behavior.

In this case, end A of the compression member has only one restaining member $A1$. Its direction cosines are $l = 0$, $m = 1$, and $n = 0$, and its stiffness with the far end hinged is $K = 3EI/L$. From this we find [Eqs. (21) and (25)] that $S_1 = S_2 = S_3 = S_6 = 0$, $S_4 = -K$, and $S_5 = S_7 = K$. Equation (23) then gives $\tan 2\gamma = 0$, whence $\gamma = 0°$ and

Table 4 Calculations for Example 4

Bar	i	l	m	n	lm	ln	mn	$n^2 - m^2$	$1 - l^2$	$1 - m^2$
$A1$	1	0	1	0	0	0	0	−1	1	0
$A2$	2	0	0	1	0	0	0	1	1	1
	$\sum_i$	...	...	...	0	0	0	0	2	1

$90°$. The first value of γ corresponds to buckling of AB in a plane perpendicular to the original plane of the structure, the second to buckling in the plane. The associated end-restraint stiffnesses, from Eq. (27), are $m = 0$ for $\gamma = 0°$ and $m = K$ for $\gamma = 90°$. These are the values one would expect for the two planes of buckling in view of the assumed ball-and-socket joint at 1. The corresponding values of P_{cr} from Table 1, are

$$P_{cr} = \pi^2 EI/L^2 \qquad \text{for} \qquad \gamma = 0°$$

$$P_{cr} = (4.36)^2 EI/L^2 \qquad \text{for} \qquad \gamma = 90°$$

It is seen that in this case the compression member will buckle out of the plane as a pin-ended column, rather than in the plane. For either type of buckling, Eq. (22) gives $\phi/\theta = 0$, indicating, as expected, that the bending rotation of end A of the compression member during buckling is unaccompanied by any twisting rotation. In the following example, the twisting rotation will be nonzero for one plane of buckling, and its significance will be readily discerned.

Example 3

The conditions of this example are the same as those of the preceding example except that the restraining bars now make an angle of 60° with respect to the compression member (see Fig. 5). The one restraining member at A has the direction cosines $l = \frac{1}{2}$, $m = (\frac{3}{4})^{1/2}$, and $n = 0$, whence $S_1 = (3^{1/2}K/4)$, $S_2 = S_3 = 0$, $S_4 = -3K/4$, $S_5 = 3K/4$, $S_6 = K/4$, and $S_7 = K$. Again, Eq. (23) yields $\gamma = 0°$ or $90°$, and Eq. (27) yields the stiffnesses $m = 0$ and $m = K$, respectively. The corresponding buckling loads are exactly the same as for Example 2.

Of course, it is expected that the structures of Figs. 4 and 5 have the same critical load for buckling in the plane ($\gamma = 90°$); however it is surprising, at first glance, to find that the critical loads for out-of-plane buckling ($\gamma = 0°$) are also the same. The explanation lies in the value of ϕ/θ for out-of-plane buckling. Substituting $\gamma = 0°$ in Eq. (22), we find that out-of-plane buckling of member AB in Fig. 5 occurs with $\phi/\theta = (\frac{1}{3})^{1/2}$, or, from Eq. (10), $\boldsymbol{\omega} = \theta[\mathbf{i}\ (\frac{1}{3})^{1/2} + \mathbf{j}]$. Thus the instantaneous axis of rotation of joint A coincides with the restraining member $A1$. Member $A1$ merely swivels in its ball-and-socket joint without flexing. This explains why the out-of-plane buckling load of the compression member of Fig. 5 is that of a pin-ended column.

Example 4

The bar group of Fig. 6 now is considered, in which all bars have the same length L, Young's modulus E, and moment of inertia I. The critical load P_{cr} for elastic buckling of bar AB is sought. In this example, the elastic restraint at end A of the compression member is the same for rotation about the y axis as for rotation about the z axis.

The restraining members $A1$ and $A2$, being unstressed, have the following stiffnesses with far end hinged: $K_1 = K_2 = 3EI/L$, which will be denoted by the common symbol K. The calculations needed for the evaluation of S_1–S_7 are shown in Table 4. Multiplying the sums in the bottom line of Table 4 by K, we find $S_1 = S_2 = S_3 = S_4 = 0$, $S_5 = 2K$, and $S_6 = S_7 = K$. Equation (23) then gives $\tan 2\gamma = 0/0$; that is, γ is indeterminate, indicating that all planes through AB are theoretically possible planes of buckling. From Eqs. (28), $m_y = K\cos\gamma$, and $m_z = K\sin\gamma$. Equation (27) then gives $m = K$ for all γ, indicating that there is no preferred buckling plane. The buckling load corresponding to $m = K = 3EI/L$ is found from Table 1 to be $P_{cr} = (4.35)^2 EI/L^2$.

Conclusion

An approximate method has been presented for checking the stability of a compression member in a rigid-joint welded-tubular truss that is symmetrical about a plane perpendicularly bisecting the compression member. The method is based on the assumption that the restraining members emanating from the ends of the compression member are supported at their far ends by ball-and-socket joints.

It was shown that the compression member may, theoretically, buckle in either of two mutually perpendicular directions. Formulas were developed for determining these two directions and the end-restraint stiffness associated with each one. It was presumed that buckling would occur in that direction having the smaller value of associated end-restraint stiffness, and a table was presented that gives the buckling load as a function of the end-restraint stiffness. Comparison of this computed buckling load and the actual load in the compression member indicates whether or not the compression member is stable.

The method is applicable to space trusses and to the out-of-plane or in-plane buckling of members in plane trusses. Its use is illustrated by means of four numerical examples.

References

[1] Renton, J. D., "Stability of space frames by computer analysis," J. Struct. Div., Am. Soc. Civil Engrs. **88,** 81–103 (1962).

[2] Bleich, F., *Buckling Strength of Metal Structures* (McGraw-Hill Book Company Inc., New York, 1952).

[3] Lundquist, E. E., "Stability of structural members under axial load," NACA TN 617 (1937).

[4] Lundquist, E. E., "Principles of moment distribution applied to stability of structural members," Proc. 5th Intern. Congr. Appl. Mech. 145–150 (1938).

[5] Hoff, N. J., "Stable and unstable equilibrium of plane frameworks," J. Aeronaut. Sci. **8,** 115–119 (1941).

[6] Sandorff, P. E., "Column end-fixity vs end-restraint," J. Aerospace Sci. **26,** 759–760 (November 1959).

[7] Osgood, W. R., "Column strength of tubes elastically restrained against rotation at the ends," NACA Rept. 615 (1938).

[8] Lundquist, E. E., Rossman, C. A., and Houbolt, J. C., "A method for determining the column curve from tests of columns with equal restraints against rotation on the ends," NACA TN 903 (August 1943).

[9] Lundquist, E. E. and Kroll, W. D., "Extended tables of stiffness and carry-over factor for structural members under axial load," NACA ARR 4 B 24 (February 1944); also NACA WR L-255.

[10] "Strength of metal aircraft elements," MIL-HDBK-5, Armed Forces Supply Support Center, Fig. 2.4.2.3(c) (March 1961).

[11] Lundquist, E. T., "A method of estimating the critical buckling load for structural members," NACA TN 717 (July 1939).